Explicit, analytical solution of scaling quantum graphs

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Abstract

Based on earlier work on regular quantum graphs we show that a large class of scaling quantum graphs with arbitrary topology are explicitly analytically solvable. This is surprising since quantum graphs are excellent models of quantum chaos and quantum chaotic systems are not usually explicitly analytically solvable.

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Integrable systems are rare. Many have become textbook classics, such as the harmonic oscillator and the hydrogen atom. It is therefore always a noteworthy event when another integrable system is discovered. The purpose of this Letter is to add an important class of quantum systems to this set: quantum graphs. Quantum graphs have a long history in mathematics, chemistry and physics (see, e.g. the excellent review by P. Kuchment [1]). They have recently come to the attention of many researchers as models of quantum chaos [2–5]. It may therefore be surprising that a large and important subset of quantum graphs, scaling quantum graphs, is explicitly integrable in the form $E_n = f(n)$, where n is an integer, E_n are the energy levels of the graph and f is a function that can be constructed explicitly. Because of the quantum chaos connection it already caused quite a stir when about a year ago explicit solutions of a sub-class of quantum graphs, regular quantum graphs, were discovered [6,7]. This is so, because for any given n the energy level E_n is individually computable, without knowledge of the other quantum levels. Thus the counting index n is a quantum number which is generally believed not to exist in a quantum system such as quantum graphs that show so many features of quantum chaos [8,9]. While in the case of regular quantum graphs one might shrug off the existence of explicit solutions by pointing to the special, non-generic nature of regular quantum graphs (only simply connected, linear and circular graphs have so far been identified as members of the set of regular quantum graphs), this is now no longer possible. Thus the results reported in this Letter are a substantial, qualitative step forward with profound implications for quantum graph theory and quantum chaos. Focussing on scaling quantum graphs [6,7] excluding graphs with tunneling and bound states for convenience (tunneling and bound states are discussed in [10]), we will prove below that these quantum graphs, no matter how complex their topology, are explicitly solvable analytically. We emphasize that *scaling* quantum graphs [6,7] represent a huge class of graphs, far larger than and *including* the standard, undressed graphs primarily studied in the literature [1–5]. The class of quantum graphs for which we prove the existence of explicit solutions comprise quantum graphs with arbitrary topology, arbitrary scaling potentials on their bonds and arbitrary scaling δ-functions on their vertices.

It has been shown before [6,7] that the spectral function of scaling quantum graphs is given by

$$g^{(0)}(k) = \cos(S_0 k + \varphi_0) - \sum_{j=1}^{N} a_j \, \cos(S_j k + \varphi_j), \tag{1}$$

where S_0 is the action length of the graph, $S_j < S_0$ are certain combinations of the graph actions, N is the number of different graph action combinations in (1), φ_0, φ_j are constant phases and a_j are constant amplitudes. The roots $k_n^{(0)} = \sqrt{E_n}$ of the spectral equation

$$g^{(0)}(k_n^{(0)}) = 0 \tag{2}$$

are the wave numbers of the eigenenergies E_n of the graph. It was shown in [7] and put on a solid mathematical foundation in [11] that (2) is explicitly solvable in the form $k_n^{(0)} = \ldots$ for regular quantum graphs [6,7,11] which satisfy

$$\sum_{j=1}^{N} |a_j| < 1. \tag{3}$$

The vast majority of quantum graphs does not satisfy the regularity condition (3). Up to now this was believed to be the hedge that makes general quantum graphs resilient against explicit solution and thus preserves one of the central tenets of quantum chaos theory, i.e. the loss of quantum numbers [8,9]. However, there is a way to solve all scaling quantum graphs, even if they don't fulfill (3), by constructing a chain of spectral functions terminating with a function that does satisfy (3), and then making full use of (3) and the associated mathematical machinery developed for regular quantum graphs [11]. Thus the theory of regular quantum graphs provides us with a powerful point of departure for solving the more general problem of all scaling quantum graphs.

Taking the m'th derivative of (1) and dividing by S_0^m we obtain

$$g^{(m)}(k) = \cos(S_0 k + \varphi_0 + m\pi/2) - \sum_{j=1}^{N} b_j^{(m)} \, \cos(S_j k + \varphi_j + m\pi/2), \tag{4}$$

where $b_j^{(m)} = a_j (S_j/S_0)^m$. Since $S_j < S_0$, there always exists an M such that $\sum |b_j^{(M)}| < 1$, i.e. for $m = M$ (4) satisfies the regularity condition (3) and the roots $k_q^{(M)}$ of $g^{(M)}(k) = 0$ are explicitly computable via explicit analytical, convergent periodic-orbit expansions as shown in [6,7] and proved rigorously in [11].

Having bootstrapped the roots of (1) at the level M by taking the M'th derivative of (1), we now have to go backwards to the levels $M-1$, $M-2$, ..., until we reach level 0 and possess $k_n^{(0)}$ explicitly. Retracing our steps is indeed possible by making use of so-called root-separators $\hat{k}_n$, which play an important role in the theory of regular quantum graphs [11]. A separator $\hat{k}_n$ separates roots number $n-1$ and number n from each other. Because the theory of regular quantum graphs applies to $g^{(M)}(k)$, we know the separators $\hat{k}_q^{(M)}$. To go one level up to $M-1$, we need to know the separators $\hat{k}_r^{(M-1)}$ which separate roots number $r-1$ and number r from each other. It turns out [10] that the extrema of $g^{(M-1)}(k)$ are root separators on the level $M-1$. But the location of the extrema of $g^{(M-1)}(k)$ are the zeros of $g^{(M)}(k)$, which we know. With the knowledge of the root separators $\hat{k}_r^{(M-1)}$ we now compute the zeros $k_r^{(M-1)}$ of $g^{(M-1)}(k)$:

$$k_r^{(M-1)} = \int_{\hat{k}_{r-1}^{(M-1)}}^{\hat{k}_r^{(M-1)}} k\,|g^{(M)}(k)|\,\delta(g^{(M-1)}(k))\,dk. \tag{5}$$

It is important to realize that (5) is not just a formal solution, but yields $k_r^{(M-1)}$ explicitly, analytically, even numerically, if we wish, to any desired accuracy. This is so since both $g^{(M-1)}(k)$ and the root separators $\hat{k}_r^{(M-1)}$ are known explicitly. It is furthermore important to realize that the "trick" (5) does not generally work for other quantum systems, even if their spectral equation g is known. This is so because the root separators $\hat{k}_n$, known and explicitly constructable in the case of quantum graphs, are not in general known for other quantum systems. So for more general quantum (chaotic) systems the missing separators is indeed a hedge that, so far, protects them against explicit solution.

Our retracing process from levels M to $M-1$ can now be continued until we reach level 0. This completes our proof that all scaling quantum graphs can be solved analytically in closed form.

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